

# A NUMERICAL EXPERIMENT ON BURGER'S EQUATION

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**Abstract**—There are many equations in mathematics which are used in our practical life. Burger's equation is one of them which is a good simplification of Navier-Stokes equation where the velocity is one spatial dimension and the external force is neglected in absence of pressure gradient. This equation is used to analyze traffic congestion and acoustics. It occurs in various areas of applied mathematics, such as modeling of various problems in fluid dynamics and traffic flow etc. Due to the complexity of the analytical solution, one needs to use numerical methods to solve this equation. For this we investigate finite difference method for Burger's equation and present an explicit central difference scheme. We implement the numerical by computer programming for artificial initial and boundary data and verify the qualitative behavior of the numerical solution of burger's equation.

**Index Terms**— Burger's equation, Navier-Stokes equation, Cauchy problem, Inviscid fluid, Viscous fluid, Finite difference schemes, Analytical solution, Numerical solution.

## 1 INTRODUCTION

THE one-dimensional Burger's equation [1] has received an enormous amount of attention since the studies by J.M. Burger's [2] in the 1940's, principally as a model problem of the interaction between nonlinear and dissipative phenomena. Even though it is a simplest case study' which in many setting is not realistic, it has been important in wide range of mathematical problems, from hydrodynamics to geometry. It is now realized that Burger's equation was used by a number of scientists before its re-introduction by Burgers, for example see H. Bateman [3] and A.R. Forsyth[4]. It is now known that it was first introduced by Bateman [3] in 1915 who found its steady solutions, descriptive of certain viscous flows. It was later proposed by Burgers [4] as one of a class of equations describing mathematical models of turbulence and due to the extensive work of Burger it is now known as Burger's equation. It involves series solution that converges very slowly for small values of the viscosity constant [5]. Many authors Cole, J.D [6], Mittal R.C and Singhal P [7], Caldwell, J., P. Wanless and A.E. Cook [8] have discussed the numerical solution of Burger's equation using Finite Difference Methods and Finite Element Methods.

The applications of Burger equation are demonstrated in the modeling of water in unsaturated soil, dynamics of soil water, statistics of flow problems, mixing and turbulent diffusion, cosmology and seismology [9, 10, 11].

In the content of gas dynamics, it was discussed by Hopf and Cole. They also illustrated independently that the Burger's equation can be solved exactly for an arbitrary initial condition. Benton and Platzman have surveyed the analytical solu-

tions of the one dimensional Burgers equation. It can be considered as a simplified form of the Navier-Stokes equation due to the form of non-linear convection term and the occurrence of the viscosity term.

In order to understand the non-linear phenomenon of the Navier-Stokes equation, one needs to study Burger's equation analytically and numerically as well. Many works has been appeared in the last several years e.g. [12], [13]etc.

In this paper, we present the analytical solution of one-dimensional Burger's equation as an initial value problem in infinite spatial domain and some numerical methods for solution of Burger's equation as an initial boundary value problem.

## 2 BURGER'S EQUATION

Burger's equation is a fundamental partial differential equation from fluid mechanics. It occurs in various areas of applied mathematics, such as modeling of gas dynamics and traffic flow. It is named for Johannes Martinus Burgers (1895-1981). The Burger's equation was known to Forsyth (1906) and had been discussed by Bateman (1915). Due to extensive works of Burgers (1948) it is known as Burger's equation. It is a nonlinear equation for which exact solutions are known and is therefore important as a benchmark problem for numerical methods. Burger's equation is a good simplification of Navier-Stokes equation where the velocity is in one spatial dimension and the external force is neglected and without any pressure gradient. The Burger's equation has been used to test and investigate the numerical method for Navier-Stokes equation. This equation is used to analyze traffic congestion and acoustics.

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### 2.1 Derivation of Burger's equation

The Navier-Stokes equation is given by

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla) \mathbf{q} = \mathbf{F} - \frac{1}{\rho} \nabla p + \nu \Delta \mathbf{q}$$

The 1D form of this equation is written as

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = F_x - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial x^2}$$

Now in absence of external force and no pressure gradient the above equation takes the form

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( \frac{1}{2} u^2 \right) = \nu \frac{\partial^2 u}{\partial x^2}$$

This equation is known as burger's equation. It was named for Johannes Martinus Burger's (1895 – 1981). This is non-linear 2<sup>nd</sup> order partial differential equation.

Burger's equation is a fundamental partial differential equation from fluid mechanics. It occurs in various areas of applied mathematics, such as modeling of traffic flow and gas dynamics etc.

When  $\nu \rightarrow 0$ , Burger's equation becomes the inviscid burger's equation and written as

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$$

This is 1<sup>st</sup> order Quasi-linear partial differential equation. This is prototype for equation for which the solution can develop discontinuous (shock wave). The previous equation is the convection form of the Burger's equation.

### 2.2 Solution of inviscid Burger's equation

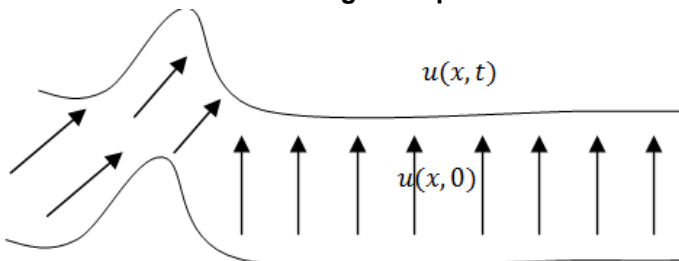


Fig. 1. characteristics and solution for Burger's equation (small t)

The inviscid Burger's equation is first order partial differential equation. Its solution can be constructed by the method of characteristics.

Consider the inviscid equation in the above figure with smooth initial data. For small time, a solution can be constructed by following characteristics.

Notice that figure looks like an advection equation, but with the advection velocity  $u$  equal to the value of the advected

quantity.

The characteristics satisfy  $x'(t) = u(x(t), t)$  and each characteristic  $u$  is constant, since

$$\begin{aligned} \frac{\partial q}{\partial t} u(x(t), t) &= \frac{\partial}{\partial t} u(x(t), t) + \frac{\partial}{\partial t} u(x(t), t) x'(t) \\ &= u_t + uu_x = 0 \end{aligned}$$

Moreover  $u$  is constant on each characteristic, the slope  $x'(t)$  is constant and so the characteristics are straight lines, determined by the initial data (figure)

If the initial data is smooth then this can be used to determine the solution  $u(x, t)$  for small enough  $t$  that characteristics do not cross. For each  $(x, t)$  we can solve the equation

$$x = \xi + u(\xi, 0)t$$

For  $\xi$  and then

$$U(x, t) = u(\xi, 0)$$

There is an implicit relation that determines the solution of the inviscid burger's equation provided characteristics do not intersect. If the characteristics do intersect, then a classical solution to the PDE does not exist.

The viscous Burger's equation can be linearized by the Cole-Hopf substitution

$$U = -2\nu \frac{1}{\phi} \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \phi}{\partial x^2}$$

which turns into the diffusion equation

$$\frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \phi}{\partial x^2}$$

That allows one to solve an initial value problem:

$$U(x, t) = -2\nu$$

$$\ln \left\{ (4\nu t)^{\frac{1}{2}} \int_{-\infty}^{\infty} \exp \left[ -\frac{(x-x')^2}{4\nu t} - \frac{1}{2\nu} \int_0^{x'} u(x'', 0) dx'' \right] dx' \right\}$$

### 3 Analytical Solution

In this chapter, we solve 1d viscous burger's equation for initial condition in infinite space analytically by transforming to heat equation

#### 3.1 Burger's equation as an IV problem

We need to solve the following Initial value problem

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( \frac{1}{2} u^2 \right) = \nu \frac{\partial^2 u}{\partial x^2} \tag{1}$$

$$\text{With I.C. } u(x, 0) = u_0(x), \text{ for } -\infty < x < \infty \tag{2}$$

#### 3.2 The Cauchy Problem

The Cauchy problem for the Heat Equation is

$$\frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \phi}{\partial x^2} \tag{3}$$

$$\phi(x, 0) = \phi_0 = e^{-\frac{1}{2\nu} \int_0^x u_0(z) dz} \tag{4}$$

which is a pure initial value problem.

#### 3.3 The fundamental solution

In this section, we derive the fundamental solution and show how it is used to solve the above Cauchy problem.

The heat equation has a scale invariance property that is analogous to scale invariance of the wave equation or scalar conservation laws, but the scaling is different.

Let  $a > 0$  be a constant. Under the scaling  $x \rightarrow ax, t \rightarrow a^2t$  the heat equation is unchanged. More precisely, if we introduce the change of variables:  $\bar{t} = a^2t, \bar{x} = ax$ ; then the heat equation becomes

$$\frac{\partial \phi}{\partial \bar{t}} = \nu \frac{\partial^2 \phi}{\partial \bar{x}^2}$$

This scale invariance suggests that we seek solutions  $v$  depending on the similarity variable  $\frac{x^2}{t}$ , or on  $\frac{x}{\sqrt{t}}$ . However,

there is a property of the heat equation we would like to preserve in Our similarity solution, that of conservation of energy. Suppose  $\phi$  is a solution of the heat equation with the property

$$\text{that } \left| \int_{-\infty}^{\infty} \phi(x,0) dx \right| < \infty \quad \text{and} \quad \phi_x(x,t) \rightarrow 0 \quad \text{as}$$

$x \rightarrow \pm\infty$ . Then, integrating the PDE, we find

$$\frac{d}{dt} \int_{-\infty}^{\infty} \phi(x,t) dx = 0$$

So that the total heat energy is conserved:

$$\int_{-\infty}^{\infty} \phi(x,t) dx = \text{Constant} \quad (5)$$

However,

$$\int_{-\infty}^{\infty} w\left(\frac{x}{\sqrt{t}}\right) dx = t^{\frac{1}{2}} \int_{-\infty}^{\infty} w(y) dy$$

This suggests we should scale the function  $w$  by  $t^{\frac{1}{2}}$ :

$$\phi(x,t) = \frac{1}{\sqrt{t}} w\left(\frac{x}{\sqrt{t}}\right) \quad (6)$$

With the scaling, heat is conserved in the sense of (5).

Substituting (6) into the PDE (3) leads to an ODE for  $w = w(y)$ , with non-constant coefficients:

$$\nu w''(y) + \frac{1}{2} y w'(y) + \frac{1}{2} w(y) = 0 \quad (7)$$

Since this is a second order equation, we should have two independent solutions. First rewrite the ODE as

$$\nu w''(y) + \frac{1}{2} (y w(y))' = 0$$

Thus,

$$\nu w'(y) + \frac{1}{2} y w(y) = \text{const} \tan t.$$

Since we are really only seeking one solution, it is convenient to set the constant to zero, and write the solution of the homogeneous equation:

$$w(y) = A e^{\frac{y^2}{4\nu}}$$

Converting back to  $(x,t)$  with  $y = \frac{x}{\sqrt{t}}$ , we obtain the similarity solution

$$\phi(x,t) = A \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4\nu t}} \quad (8)$$

Usually, we choose a particular value of  $A$  so that constant in (5) is unity i.e.

$$1 = \frac{A}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{4\nu t}} dx = \frac{2A}{\sqrt{t}} \int_0^{\infty} e^{-\frac{x^2}{4\nu t}} dx$$

Making the variable transformation  $y = \frac{x^2}{4\nu t}$ , we have,

$$\begin{aligned} 1 &= \frac{2A}{\sqrt{t}} \int_0^{\infty} e^{-\frac{y}{\sqrt{4\nu t y}}} dy = A \sqrt{4\nu} \int_0^{\infty} e^{-y} y^{-\frac{1}{2}} dy \\ &= A \sqrt{4\nu} \sqrt{\pi} \\ \Rightarrow A &= \frac{1}{\sqrt{4\nu\pi}} \end{aligned} \quad (9)$$

For this choice of constant, we have the fundamental solution of the heat equation:

$$\phi(x,t) = \frac{1}{\sqrt{4\nu\pi t}} e^{-\frac{x^2}{4\nu t}} \quad (10)$$

### 3.4 Solution of the Cauchy problem

The fundamental solution (10) satisfies (3) for  $t > 0$ .

Now  $\phi(x-y,t)$  is a solution of (3) for all  $y$ , by translation invariance:  $x = x-y$  does not change the heat equation.

Thus,

$$\phi(x-y,t)\phi_0(y)$$

is also a solution of (3). For later reference, we note that the heat equation is invariant under time translation also.

By linearity and homogeneity of the PDE, we can also take linear combinations of solutions. This suggests that

$$\phi(x,t) = \int_{-\infty}^{\infty} \phi(x-y,t)\phi_0(y) dy \quad (11)$$

should also be a solution. Moreover, properties of  $\phi$  suggest that as  $t \rightarrow 0^+$ ,  $\phi(x,t) \rightarrow \phi_0(x)$  since  $\phi(x-y,t)$  collapses to zero away from  $y = x$ , and blows up at  $y = x$  in such a way (i.e., preserving  $\int \phi = 1$ ) that the initial condition is satisfied in the sense  $\phi(x,t) \rightarrow \phi_0(x)$  as  $t \rightarrow 0^+$ .

It is straightforward to check that the integrals for  $\phi, \phi_t, \phi_{xx}$  all converge provided  $g \in C(\mathcal{R})$  is bounded. Then

$$\phi_t = \int_{-\infty}^{\infty} \frac{\partial \phi}{\partial t}(x-y, t) \phi_0(y) dy;$$

$$\phi_{xx} = \int_{-\infty}^{\infty} \frac{\partial^2 \phi}{\partial x^2}(x-y, t) \phi_0(y) dy,$$

so that  $u$  satisfies the PDE for  $t > 0$ .

It is more complicated to check the initial condition is satisfied. We need to show  $\phi(x, 0) = \phi_0(x)$ . But  $t = 0$  is a singular point for  $\phi: \phi(x, t)$  is not defined at  $t = 0$ . To get an idea of why  $\lim_{t \rightarrow 0^+} \phi(x, t) = \phi_0$ , let's fix  $x$ .

Then, for  $\delta > 0$ ,

$$\begin{aligned} & \int_{-\infty}^{\infty} \phi(x-y, t) \phi_0(y) dy \\ &= \int_{(x-y) < \delta} \phi(x-y, t) \phi_0(y) dy + \int_{|x-y| < \delta} \phi(x-y, t) \phi_0(y) dy \\ &\approx \int_{|x-y| < \delta} \phi(x-y, t) \phi_0(y) dy \end{aligned}$$

By continuity,  $\phi_0(y) \approx \phi_0(x)$  for  $y$  near  $x$ , this explains how the first integral is approximately the final line. The second integral approaches zero as  $t \rightarrow 0^+$ , because  $\phi \rightarrow 0$  uniformly, and exponentially, away from  $y = x$  as  $t \rightarrow 0^+$ .

Finally we get the solution of the Cauchy problem described in 2.2 as follows

$$\phi(x, t) = \int_{-\infty}^{\infty} \phi(x-y, t) \phi_0(y) dy \tag{12}$$

**3.5 The analytical solution of Burger's equation is as follows**

$$u(x, t) = \frac{\int_{-\infty}^{\infty} (x-y) \exp\left[-\frac{(x-y)^2}{4vt} - \frac{1}{2v} \int_0^y u_0(z) dz\right] dy}{t \int_{-\infty}^{\infty} \exp\left[-\frac{(x-y)^2}{4vt} - \frac{1}{2v} \int_0^y u_0(z) dz\right] dy} \tag{13}$$

**3.6 Numerical evaluation of Analytical solution**

Now we are interested how our analytical solution behaves when we try to implement it numerically. In order to perform numerical estimation, we have to consider a function  $u_0$  for which the two integrations appeared in the numerator and denominator of (13) converge. We know that any bounded function does the trick. We consider the bounded periodic function  $u_0(x) = \sin x$  as initial condition and find the solution over the bounded spatial domain  $[0, 2\pi]$  at different time steps. For the above initial condition we get the following analytical solution of Burger's equation,

$$u(x, t) = \frac{\int_{-\infty}^{\infty} (x-y) \exp\left[-\frac{(x-y)^2}{4vt} + \frac{1}{2v} \cos y\right] dy}{t \int_{-\infty}^{\infty} \exp\left[-\frac{(x-y)^2}{4vt} + \frac{1}{2v} \cos y\right] dy} \tag{14}$$

For very small  $v$ , both numerator and denominator of (3.4.1) get more closed to zero or get more larger which becomes very difficult to handle. So considering the value of  $v$  arbitrarily very small, we can not perform our numerical experiment.

We consider the value of  $v$  as 0,1.

Now there is another problem of calculating the value of  $u$  near initial time.

We observe from (14) that for very small  $t$ , both numerator and denominator get much closed to zero and thus difficult to handle numerically.

**3.7 Boundary values of the Analytical solution**

In this section, we find the values of the analytical solution with initial condition  $u_0 = \sin x$  at the boundaries of the spatial domain  $[0, 2\pi]$  which in further will be used as boundary conditions when we perform numerical schemes to compare the numerical solution with corresponding analytical ones.

For initial condition  $u_0 = \sin x$ , we get the analytical solution

$$u(x, t) = \frac{\int_{-\infty}^{\infty} (x-y) \exp\left[-\frac{(x-y)^2}{4vt} + \frac{1}{2v} \cos y\right] dy}{t \int_{-\infty}^{\infty} \exp\left[-\frac{(x-y)^2}{4vt} + \frac{1}{2v} \cos y\right] dy} \tag{15}$$

as described in Section 15.

For  $x = 0$ ,

$$u(0, t) = \frac{\int_{-\infty}^{\infty} (-y) \exp\left[-\frac{y^2}{4vt} + \frac{1}{2v} \cos y\right] dy}{t \int_{-\infty}^{\infty} \exp\left[-\frac{y^2}{4vt} + \frac{1}{2v} \cos y\right] dy}$$

The function under integration sign in the numerator of is an odd function, so we must have

$$\int_{-\infty}^{\infty} (-y) \exp\left[-\frac{y^2}{4vt} + \frac{1}{2v} \cos y\right] dy = 0$$

Which implies that  $u(0, t) = 0$ .

Now, for  $x = 2\pi$ , we have,

$$u(2\pi, t) = \frac{\int_{-\infty}^{\infty} (2\pi-y) \exp\left[-\frac{(2\pi-y)^2}{4vt} + \frac{1}{2v} \cos y\right] dy}{t \int_{-\infty}^{\infty} \exp\left[-\frac{(2\pi-y)^2}{4vt} + \frac{1}{2v} \cos y\right] dy}$$

Making the variable change  $z = 2\pi - y$ , we have,

$$u(2\pi, t) = \frac{\int_{-\infty}^{\infty} z \exp\left[-\frac{z^2}{4vt} + \frac{1}{2v} \cos y\right] dy}{t \int_{-\infty}^{\infty} \exp\left[-\frac{z^2}{4vt} + \frac{1}{2v} \cos y\right] dy}$$

The function under integral sign in the numerical of is an odd function, so we have,

$$\int_{-\infty}^{\infty} z \exp\left[-\frac{z^2}{4vt} + \frac{1}{2v} \cos y\right] dy = 0$$

which implies that  $u(2\pi, t) = 0$ .

So we have the boundary values

$$u(0, t) = u(2\pi, t) = 0$$

#### 4 Numerical Solution

In this chapter, we present some numerical methods to solve the Burger's equations as an initial boundary value problem.

We use explicit and implicit finite difference schemes to solve Burger's equation and then try to proceed in a different way using Cole-Hopf transformation. We solve our C-H transformed heat equation with Neumann boundary conditions using both of explicit and implicit finite difference schemes for heat equation.

##### 4.1 Explicit Upwind Difference Scheme of Burger's equation for inviscid fluid

Consider the inviscid Burger's equation as an initial boundary value problem

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left(\frac{1}{2} u^2\right) = 0; \quad t > 0, x \in (a, b) \quad (16)$$

I.C  $u(0, x) = u_0(x)$

B.C  $u(t, a) = u_a(t), u(t, b) = u_b(t)$

From (1) we have

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0 \quad (17)$$

Now we get the explicit upwind difference scheme for this initial boundary value problem using forward difference for time derivative

$$\frac{\partial u}{\partial t} \approx \frac{u_i^{n+1} - u_i^n}{\Delta t} = 0$$

and the backward difference for spatial derivative

$$\frac{\partial w}{\partial x} \approx \frac{w_i^n - w_{i-1}^n}{\Delta x} = 0$$

For  $w = \frac{1}{2} u^2$  we find  $\frac{1}{2} \frac{\partial}{\partial x} (u^2) \approx \frac{(u_i^n)^2 - (u_{i-1}^n)^2}{2\Delta x} = 0$

Then from (8) we get

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{(u_i^n)^2 - (u_{i-1}^n)^2}{2\Delta x} = 0$$

or,  $u_i^{n+1} = u_i^n - \frac{\Delta t}{2\Delta x} [(u_i^n)^2 - (u_{i-1}^n)^2]$

which is the explicit upwind difference scheme of inviscid Burger's equation.

##### 4.2 Explicit Central Difference Scheme of Burger's equation for inviscid fluid

Consider the inviscid Burger's equation as an initial boundary value problem

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left(\frac{1}{2} u^2\right) = 0; \quad t > 0, x \in (a, b) \quad (18)$$

I.C  $u(0, x) = u_0(x)$

B.C  $u(t, a) = u_a(t), u(t, b) = u_b(t)$

Now we take the forward difference formula for time derivative

$$\frac{\partial u}{\partial t} \approx \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

And the central difference formula for spatial derivative we know

$$\frac{\partial w}{\partial x} \approx \frac{w_{i+1}^n - w_{i-1}^n}{2\Delta x}$$

For equation (1)  $w = \frac{1}{2} u^2$  then

$$\frac{\partial}{\partial x} \left(\frac{1}{2} u^2\right) \approx \frac{1}{2} \frac{\partial}{\partial x} (u^2) \approx \frac{(u_{i+1}^n)^2 - (u_{i-1}^n)^2}{4\Delta x}$$

Then from (1) we can write

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{(u_{i+1}^n)^2 - (u_{i-1}^n)^2}{4\Delta x}$$

Or,  $u_i^{n+1} = u_i^n - \frac{\Delta t}{4\Delta x} [(u_{i+1}^n)^2 - (u_{i-1}^n)^2]$

Consider as  $u_i^n = \frac{u_{i-1}^n - u_{i+1}^n}{2}$

Then we can write,

$$u_i^{n+1} = \frac{u_{i-1}^n + u_{i+1}^n}{2} - \frac{\Delta t}{4\Delta x} [(u_{i+1}^n)^2 - (u_{i-1}^n)^2]$$

which is the Lux-Friedrich scheme.

##### 4.3 Explicit Central Difference Scheme of Burger's equation for viscous fluid

Consider the Burger's equation as an initial boundary value problem



$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( \frac{1}{2} u^2 \right) = v \frac{\partial^2 u}{\partial x^2}; x \in (a,b), t \in (0,T) \quad (18)$$

I.C.  $u(0, x) = u_0(x)$

B.C.  $u(t, a) = u_a(t), u(t, b) = u_b(t)$

For equi-distant grid, with temporal step-size  $\Delta t$  and spatial step-size  $\Delta x$

The discretization of  $\frac{\partial u}{\partial t}$  is obtained by forward difference formula

$$\frac{\partial u}{\partial t} \approx \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

The discretization of  $\frac{\partial}{\partial x} \left( \frac{1}{2} u^2 \right)$  is obtained by backward difference formula

$$\frac{\partial}{\partial x} \left( \frac{1}{2} u^2 \right) = \frac{1}{2} \frac{\partial}{\partial x} (u^2) \approx \frac{(u_i^n)^2 - (u_{i-1}^n)^2}{2\Delta x}$$

The discretization of  $\frac{\partial^2 u}{\partial x^2}$  is obtained by central difference approximation

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

Using the above approximations in (1) we have

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{(u_i^n)^2 - (u_{i-1}^n)^2}{2\Delta x} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

Or,

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{4\Delta x} \left[ \frac{1}{2} (u_i^n)^2 - (u_{i-1}^n)^2 \right] + v \frac{\Delta t}{\Delta x^2} (u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

which is called the Explicit Central Difference Scheme for viscous Burger's equation.

#### 4.4 Numerical formulation of Burger's equation

Our problem is to solve the following IBV problem

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = v \frac{\partial^2 u}{\partial x^2}, -a \leq x \leq b, 0 \leq t \leq T$$

With I.C.  $u(x,0) = u_0(x)$  (19)

and B.C.  $u(a,t) = u_1(t)$  (20)

and  $u(b,t) = u_b(t)$  (21)

We discretize the  $x-t$  plane by choosing a mesh width  $h \equiv \Delta x$  and a time step  $k \equiv \Delta t$ , and define the discrete mesh points  $(x_j, t_n)$  by,

$$x_i = a + ih, i = 0, 1, \dots, M \quad (22)$$

$$\text{and } t_n = nk, n = 0, 1, \dots, N \quad (23)$$

Where,

$$M = \frac{b-a}{h} \text{ and } N = \frac{T}{k}$$

Now we are interested to solve the above IBVP numerically.

#### 4.5 Explicit finite difference scheme

To obtain an explicit finite difference scheme, we discretize

$\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}$  and  $\frac{\partial^2 u}{\partial x^2}$  at any discrete point  $(x_i, t_n)$  as follows:-

$$\frac{\partial u}{\partial t} \approx \frac{u_i^{n+1} - u_i^n}{k} \text{ (First order Forward difference formula)}$$

$$\frac{\partial u}{\partial x} \approx \frac{u_{i+1}^n - u_{i-1}^n}{2h} \text{ (First order Central difference formula)}$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} \text{ (Second order Centered difference formula)}$$

Inserting the above formulas, the discrete version of the viscid Burger's equation formulates the second order finite difference scheme of the form

$$\frac{u_i^{n+1} - u_i^n}{k} + \frac{u_i^n}{2h} (u_{i+1}^n - u_{i-1}^n) = v \times \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}$$

Or,

$$u_i^{n+1} = u_i^n - \frac{ku_i^n}{2h} (u_{i+1}^n - u_{i-1}^n) + \frac{vk}{h^2} (u_{i+1}^n - 2u_i^n + u_{i-1}^n) \quad (24)$$

which is the explicit finite difference scheme for the IBVP.

#### 4.6 Numerical implementation

Now we implement the numerical finite difference scheme by computer programming and perform numerical simulation as described below.

In implementation of our scheme, we consider the spatial domain  $[0; 2]$  and

the maximum time step  $T = 5$ ;

We consider the initial condition

$$u(x,0) = u_0(x) = \sin x \quad (25)$$

and the Homogeneous Dirichlet boundary conditions

$$u(0,t) = 0 = u(2\pi,t) \quad (26)$$

For  $v = .1$ , we get the stability condition,

$$\frac{h}{2} \leq 0.1 \leq \frac{h^2}{2k}$$

$$\text{i.e } h \leq .2 \text{ and } k \leq \frac{h^2}{0.2} \quad (27)$$

For  $h = .1$ , we have  $k = .05$ . We consider  $k = .01$ ;

Figure 3.2.2: Solution of Burger's equation using explicit finite difference scheme at different time steps with  $v = .01; \Delta x = .02; \Delta t = .02$

Clearly, for  $v = .01$ , shock occurs as time grows in the middle of the graph.

### 5 Numerical Experiment and Results

We develop a computer program (code) and implement the explicit central difference scheme for Burger's equation.

#### 5.1 Data Insert

We implement the explicit central difference for numerical experiment for the Burger's equation. We implement the scheme for initial and boundary data verify the qualitative behavior of the of velocity and viscosity of the viscous Burger's equation. We choose different value of  $v$  for this.

#### 5.2 Results

To test the accuracy of the implementation of the numerical scheme for the viscous Burger's equation, we discuss our experiment and results are given below:

We perform the numerical experiment for the equation  $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} (\frac{1}{2} u^2) = v \frac{\partial^2 u}{\partial x^2}$ . We use the initial value which satisfies stability condition.

Now we implement the numerical finite central difference scheme of viscous Burger's equation by computer programming and perform numerical simulation as describe below:

In implement of our scheme, we consider the spatial domain  $[0, 2\pi]$  and the maximum time space  $T=5$ .

We consider the initial condition

$$u(x,0) = u_0(x) = \sin x$$

And the Dirichlet boundary condition  $u(0,t) = 0 = u(2\pi,t)$

For  $h = 0.002$  and  $v = 0.01$ , we get the stability condition,

$$\frac{.002}{2} \max \{u_i^0\} \leq v$$

But since the initial condition is  $u_0(x) = \sin x$ , so we must have,

$$\max \{u_i^0\} = 1$$

And stability condition becomes,

$$\frac{.002}{2} \leq v$$

i.e.  $0.001 \leq v$

for  $v = 1$  we get  $1 \leq \frac{0.002^2}{2 * k} 1 \leq -$  i.e.  $k \leq 0.000002$

For  $h = 0.002, k = 0.000002$ . We consider  $v = 0.01, 0.1, 0.3, 0.5, 1$ .

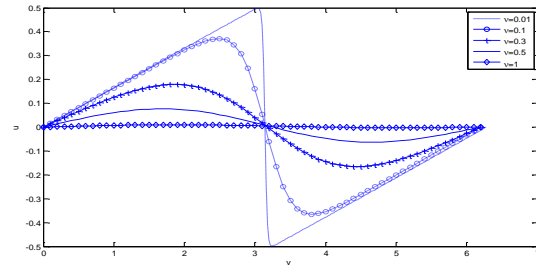


Fig. 2. Solution of Viscous Burger's equation using explicit finite difference scheme at different viscosity  $v$  with  $\Delta x=0.002, \Delta t = 0.000002$ .

After calculating stability condition we consider  $v = 0.3, 0.6, 0.9, 1.2, 1.5, 3.0$  for  $h = 0.1$  and  $k = 0.0016$  we get the following figure

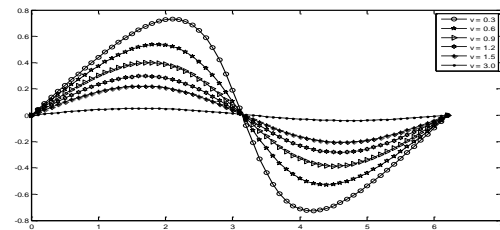


Fig. 3. Solution of viscous Burger's equation using explicit finite difference scheme at different viscosity  $v$  with  $\Delta x=0.1, \Delta t = 0.0016$ .

After calculating stability condition we consider  $v = 3.3, 3.6, 3.9, 4.2, 4.5, 6.0$  for  $h = 0.1$  and  $k = 0.0008196$ , we get the following fig. 4.

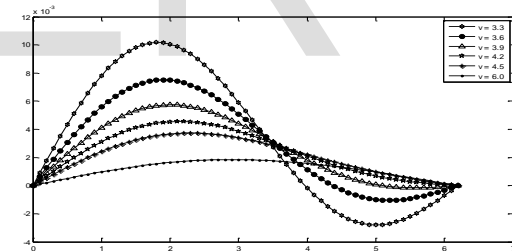


Fig. 4. Solution of viscous Burger's equation using explicit finite difference scheme at different viscosity  $v$  with  $\Delta x=0.1, \Delta t = 0.0008196$ .

After calculating stability condition we consider  $v = 6.3, 6.6, 6.9, 7.2, 7.5, 9.0$  for  $h = 0.1$  and  $k = 0.000555$ , we get the following fig 5.

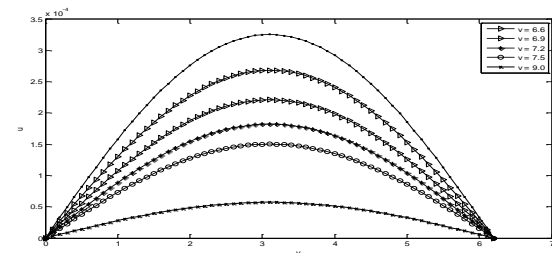


Fig. 5. Solution of viscous Burger's equation using explicit finite difference scheme at different viscosity  $v$  with  $\Delta x=0.1, \Delta t = 0.000555$ .

After calculating stability condition we consider  $v = 9.5, 10.5, 11.5, 12.5, 13.5, 15.0$  for  $h = 0.1$  and  $k = 0.000333$ , we get the

following fig. 6.

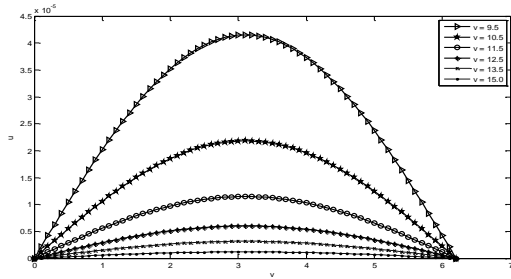


Fig. 6. Solution of viscous Burger's equation using explicit finite difference scheme at different viscosity  $\nu$  with  $\Delta x=0.1$ ,  $\Delta t=0.000333$ .

By observing above figure we can say when the viscosity  $\nu$  tends to larger then the velocity tends to smaller and when the viscosity  $\nu$  tends to smaller then the velocity tends to larger.

i.e. when  $\nu \rightarrow \infty$ , then  $u \rightarrow 0$

and when  $\nu \rightarrow 0$ , then  $u \rightarrow \infty$ .

## 6 Conclusions

Burger's equation is one of the interesting and implemented equations in our practical life for both viscous and inviscid fluid. In this paper, we have considered Burger's equation is fundamental partial differential equation from fluid mechanics. First we have shown derivation of Navier-Stokes equation, Burger's equation and numerical methods of Burger's equation. At last we have shown numerical result based on the explicit central difference scheme agrees with basic qualitative behavior of viscuss Burger's equation. In future, we try to develop better numerical ways to solve Burger's equation with initial condition and two-sided boundary conditions infinite space.

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